

A discrete computer network model with expanding dimensions*

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Abstract. Complex networks with expanding dimensions are studied, where the networks may be directed and weighted, and network nodes are varying in discrete time in the sense that some new nodes may be added and some old nodes may be removed from time to time. A model of such networks in computer data transmission is discussed. Each node on the network has fixed dimensionality, while the dimension of the whole network is defined by the total number of nodes. Based on the spectacular properties of data transmission on computer networks, some new concepts of stable and unstable networks differing from the classical Lyapunov stability are defined. In particular, a special unstable network model, called devil network, is introduced and discussed. It is further found that a variety of structures and connection weights affects the network stability substantially. Several criteria on stability, instability, and devil network are established for a rather general class of networks, where some conditions are actually necessary and sufficient. Mathematically, this paper makes a first attempt to rigorously formulate a fundamental issue of modeling discrete linear time-varying systems with expanding dimensions and study their basic stability property.

Keywords: complex network, mathematical modeling, time-varying system, dimension-varying system, stability.

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1. Introduction

Many real-world networks appear to be different but share some similar complexity in such diverse aspects as varying dimensionality, intrinsic connectivity, and complicated dynamics. In the endeavor of understanding the common forming mechanism of seemingly different networks, some recent work has already captured some essential features of various complex networks, particularly the small-world characteristic coined by Watts and Strogatz (1) and the scale-free degree distribution in an invariant power-law form discovered by Barabási and Albert (2). It is noticeable that owing to the mathematical simplification for elegance and rigor, the classical random graph theory of Erdős and Rényi (3) and the small-world model which inherits the same spirit have a fixed dimensionality for each given network. Although the scale-free model generates a growing network, the network dimension is typically restricted to be fixed as that of a single node in order to be mathematically manageable when analysis comes into play.

The main interest in this letter is to deal with a growing network with expanding dimensionality, a more realistic model for the Internet to say the least. The aim is to build a graph framework and to lay a mathematical foundation for such a model that evolves in discrete time with increasing dimensions as well as complex dynamics including directionality and weights if desirable. A typical example in point is the real data transmission on a computer network, in which the number of nodes and the connection among them are both varied in time, where the connectivity may also be directed and weighted. Indeed it is quite interesting to think of an email network in a university or a company, where one computer contaminated with reinfection-enhanced virus such as the infamous W32/Sircam will send out many copies of the virus to computers listed in its email address book, causing the system server overloaded as an immediate consequence.

A new graph model for such networks as the aforementioned computer system is established in this letter, where each node on the network has fixed dimensionality while the dimension of the whole network is defined by the total number of its nodes therefore continuously increasing. The network is characterized by a discrete linear dynamical system, where some new nodes will be added and some old nodes will be removed throughout the evolutionary process. New concepts of stable and unstable networks are introduced, which differ from the classical Lyapunov stability in several aspects. In particular, a special unstable network

model, named devil network, is discussed. It is shown that a variety of structural and connectional properties affects the network stability substantially. Several criteria on stability, instability, and the devil network are finally established, actually for a rather general class of networks, where some conditions are necessary and sufficient. The major mathematical contribution of this paper is to rigorously formulate a fundamental issue of modeling discrete linear time-varying systems with expanding dimensions and study their basic stability theory.

2. A graph model of simple computer networks

Consider an isolated local-area computer network with only one server for simplicity, assuming that at most one PC is being added to the network at a time. In the model, connections among nodes are directed and the directions may vary in time, but bi-directional data transmissions are not permitted.

In a real-world network, some new nodes may be added and some old nodes may be removed from time to time. When a node is removed from the network at some time t_0 , one treats it as an isolated node starting from t_0 . This means that those removed nodes will not have any connections with the other nodes in the network for all $t \geq t_0$, and consequently all the corresponding connection weights become 0 forever.

Assume that the computer network has n_t computers, referred to as nodes, at discrete time $t \in \mathbf{Z}^+ = \{t\}_{t=0}^\infty$. Let $x_i(t)$ be the difference in data amount between the input and output of node i , $1 \leq i \leq n_t$, at time t , whose absolute value $|x_i(t)|$ is called the storage of node i at time t . Then

$$\Delta(x)(t) = \sum_{i=1}^{n_t} x_i(t)$$

is the total difference between the input and output data of the whole network at time t . Note that this $\Delta(x)(t)$ is also the difference between the output and input data in the server, referred to as the central station, at time t , and its absolute value $|\Delta(x)(t)|$ is called the storage of the server or the whole network at time t . Only the case of finite memories is considered; namely, every computer and the server have a maximum allowable storage.

Conceptually, if the amount of data stored on a computer is less than or equal to its maximum storage and the amount of data stored on the server does not exceed its maximum at some time, then the network is running well at that moment. The network is said to be

stable if it runs well at all times. Otherwise, if the amount of data stored on some computer is larger than its maximum storage at some time, whenever in the process, the network is in a troublesome situation since it would require the server or some other computers to share the extra workloads. There is another scenario that the actual storages of all computers are less than or equal to their maximum allowable storages, but the server is overloaded, at a moment. In this latter case, the server will breakdown. Both of these two cases of the network are referred to as being unstable.

Mathematically, the above concepts of stability and instability are defined for the model as follows. Let S_i be the maximum storage of node i and M_0 the maximum storage of the server (or the whole network).

Definition 1. A network is said to be stable if there exists a positive constant $r_0 \leq M_0$ such that for all initial point $x(0) = (x_1(0), x_2(0), \dots, x_{n_0}(0))^T \in \mathbf{R}^{n_0}$ satisfying $\sum_{j=1}^{n_0} |x_j(0)| \leq r_0$, one has $|x_i(t)| \leq S_i$, $1 \leq i \leq n_t$, and $|\Delta(x)(t)| \leq M_0$, for all $t \in \mathbf{Z}^+$. Otherwise, it is said to be unstable. In particular, the network is called a devil network if it is unstable and, further, for any small positive constant $r \leq M_0$ there exists an initial point $x(0) \in \mathbf{R}^{n_0}$ with $\sum_{j=1}^{n_0} |x_j(0)| \leq r$ such that $|x_i(t)| \leq S_i$, $1 \leq i \leq n_t$, for all $t \in \mathbf{Z}^+$, and $|\Delta(x)(t_k)| \leq \alpha M_0$ for infinitely many times $t_k > 0$, $k \geq 1$, where the constant $0 < \alpha < 1$ is called a scaling parameter of the network, but $|\Delta(x)(t'_k)| > M_0$ for infinitely many times $t'_k > t_k$, $k \geq 1$.

Remark 1.

- (i) The above-introduced definition of stability for networks is different from the classical Lyapunov stability for dynamical systems (4).
- (ii) In the above definition of the devil network, the constant α , $0 < \alpha < 1$, is determined by some specific requirements on the network. For example, one may choose $\alpha = 1/2$ in the data transmission model of the computer network. If $|x_i(t)| \leq S_i$, $1 \leq i \leq n_t$, for all $t \in \mathbf{Z}^+$, and $|\Delta(x)(t_k)| \leq M_0/2$ for infinitely many times t_k , then each computer runs very well at any time and the whole network works fine at all t_k . But, in the case of $|\Delta(x)(t'_k)| > M_0$ with $t'_k > t_k$, the network would be in a rapidly changing troublesome situation (devil behaviors) after $t'_k - t_k$, $k \geq 1$.
- (iii) The condition $t'_k > t_k$, $k \geq 1$, in the definition of devil network, is not restrictive. Since $\{t_k\}_{k=1}^{\infty}$ and $\{t'_k\}_{k=1}^{\infty}$ are both infinite sequences, one can easily choose suitable

$t'_{k'}$ satisfying $t'_{k'} > t_k$, $k' > k \geq 1$.

Define the dimension of the network at time t be equal to n_t , the total number of all the nodes in the network at time t . Ignoring nonlinear factors, simply assume that $x(t) = (x_1(t), x_2(t), \dots, x_{n_t}(t))^T \in \mathbf{R}^{n_t}$ satisfies the following discrete linear system:

$$x(t+1) = A(t)x(t), \quad t \in \mathbf{Z}^+, \quad [1]$$

where $A(t) = (a_{ij}(t))$ is the coupling matrix of the network, which is an $n_{t+1} \times n_t$ matrix. Given any initial point $x(0) \in \mathbf{R}^{n_0}$, the solution $x(t)$ of system **1** can be written as

$$x(t+1) = D(t)x(0), \quad t \in \mathbf{Z}^+, \quad [2]$$

where

$$D(t) = A(t)A(t-1) \cdots A(0). \quad [3]$$

Consider the following special case of the above network model in the rest of this section, where the matrix $A(t)$ has entries taken from the triple $\{-1, 0, 1\}$ for all $t \in \mathbf{Z}^+$, which is called a T-matrix:

(H₁) The matrix $A(t)$ of system **1** is an $n_{t+1} \times n_t$ T-matrix. Assume that its entries $a_{ij}(t)$, $i \neq j$, are evaluated at t in the following way: $a_{ij}(t) = -1$ if node i sends data to node j ; $a_{ij}(t) = 1$ if node i receives data from node j ; and $a_{ij}(t) = 0$ if there is no data transmission between nodes i and j . It is natural to set $a_{ii}(t) = 0$. Clearly, the matrix $A(t)$ is antisymmetrical:

$$a_{ij}(t) = -a_{ji}(t), \quad t \in \mathbf{Z}^+.$$

(H₂) The network initially has two nodes at $t = 0$, i.e., $n_0 = 2$, and the number of nodes in the network increases by one at a time; that is, $n_t := t + 2$ for $t > 0$. The new node does not send or receive any data from all the old nodes at time t ; that is,

$$a_{t+3,j}(t) = 0, \quad 1 \leq j \leq t+2, \quad t \in \mathbf{Z}^+.$$

(H₃) The matrix $A(t)$ is in the following form:

$$A(0) = \begin{pmatrix} J \\ 0 \end{pmatrix}, \quad A(2t) = \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}, \quad t \geq 1, \quad [4]$$

$$A(2t+1) = \begin{pmatrix} -J & B(2t+1) \\ -B^T(2t+1) & C(2t+1) \\ 0 & 0 \end{pmatrix}, \quad t \in \mathbf{Z}^+, \quad [5]$$

where

$$B(2t+1) = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1,2t+1} \\ b_{21} & b_{22} & \cdots & b_{2,2t+1} \end{pmatrix} (2t+1)$$

is a $2 \times (2t+1)$ T-matrix, $C(2t+1)$ is a $(2t+1) \times (2t+1)$ antisymmetrical T-matrix, with zero block-matrices in compatible dimensions, and

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

It is clear that J is a 2×2 antisymmetrical matrix and satisfies $J^2 = -I_2$, where I_2 is the 2×2 identity matrix.

It follows from (H₃) by induction that

$$D(2t) = \begin{pmatrix} J \\ 0 \end{pmatrix}, \quad D(2t+1) = \begin{pmatrix} I_2 \\ -B^T(2t+1)J \\ 0 \end{pmatrix}, \quad t \in \mathbf{Z}^+. \quad [6]$$

For any initial point $x(0) = (a, b)^T \in \mathbf{R}^2$, it follows from **2** and **6** that the corresponding solution $x(t)$ can be written as

$$x(2t+1) = \begin{pmatrix} -b \\ a \\ 0 \end{pmatrix}; \quad x(2t+2) = \begin{pmatrix} a \\ b \\ u(2t+2) \\ 0 \end{pmatrix}, \quad t \in \mathbf{Z}^+, \quad [7]$$

where

$$\begin{aligned} u(2t+2) &= -B^T(2t+1)Jx(0) \\ &= (b_{11}b - b_{21}a, b_{12}b - b_{22}a, \dots, b_{1,2t+1}b - b_{2,2t+1}a)^T(2t+1). \end{aligned}$$

Since $b_{ij}(2t+1) \in \{-1, 0, 1\}$, it follows from **7** that for all $t \in \mathbf{Z}^+$ and for all $1 \leq j \leq n_t$,

$$|x_j(t)| \leq |a| + |b|. \quad [8]$$

In addition, it follows from **7** that

$$\begin{aligned} \Delta(x)(2t+1) &= -b + a, \\ \Delta(x)(2t+2) &= a\left(1 - \sum_{i=1}^{2t+1} b_{2i}(2t+1)\right) + b\left(1 + \sum_{i=1}^{2t+1} b_{1i}(2t+1)\right), \end{aligned} \quad [9]$$

which imply that

$$\begin{aligned} |\Delta(x)(2t+1)| &\leq |b| + |a|, \\ |\Delta(x)(2t+2)| &\leq |a| \left| 1 - \sum_{i=1}^{2t+1} b_{2i}(2t+1) \right| + |b| \left| 1 + \sum_{i=1}^{2t+1} b_{1i}(2t+1) \right|, \quad t \in \mathbf{Z}^+. \end{aligned} \tag{10}$$

As noted in Remark 1, for this computer network model, one may choose the constant $\alpha = 1/2$ in Definition 1. The discussion on the stability of system **1** with this choice is divided into the following two cases:

Case I. Suppose that the two sequences $\left\{ \sum_{i=1}^{2t+1} b_{1i}(2t+1) \right\}_{t=0}^{\infty}$ and $\left\{ \sum_{i=1}^{2t+1} b_{2i}(2t+1) \right\}_{t=0}^{\infty}$ are bounded. Then, for any $(2t+1) \times (2t+1)$ antisymmetrical T-matrix $C(t)$, the network is stable.

In fact, by the assumption there exists a positive constant γ such that

$$\left| 1 - \sum_{i=1}^{2t+1} b_{2i}(2t+1) \right|, \left| 1 + \sum_{i=1}^{2t+1} b_{1i}(2t+1) \right| \leq \gamma, \quad t \in \mathbf{Z}^+.$$

Denote

$$C_0 := \inf \{ S_j : 1 \leq j \leq n_t, t \in \mathbf{Z}^+ \} \tag{11}$$

and only consider the situation where $C_0 > 0$ in the following.

For any initial value $x(0) = (a, b)^T$ with

$$|a| + |b| \leq \min \{ C_0, M_0, M_0/\gamma \},$$

it follows from **8** and **10** that the solution $x(t)$ satisfies

$$|x_j(t)| \leq C_0 \leq S_j, \quad 1 \leq j \leq n_t, \quad |\Delta(x)(t)| \leq M_0, \quad t \in \mathbf{Z}^+. \tag{12}$$

Hence, the network is stable.

Case II. For any $(2t+1) \times (2t+1)$ antisymmetric T-matrix $C(t)$, the model is a devil network if and only if at least one of the two sequences $\left\{ \sum_{i=1}^{2t+1} b_{1i}(2t+1) \right\}_{t=0}^{\infty}$ and $\left\{ \sum_{i=1}^{2t+1} b_{2i}(2t+1) \right\}_{t=0}^{\infty}$ is unbounded.

The necessity follows from the conclusion of Case I.

To show the sufficiency, without loss of generality, suppose that $\left\{ \sum_{i=1}^{2t+1} b_{1i}(2t+1) \right\}_{t=0}^{\infty}$ is unbounded. Then, for any $a \in \mathbf{R}$ with $0 < |a| \leq \min \{ C_0, M_0/2 \}$, there exist infinitely many $t_k \geq 1, k \geq 1$, such that

$$\left| 1 - \sum_{i=1}^{2t_k+1} b_{2i}(2t_k+1) \right| > M_0/|a|.$$

Consequently, by **8** and **9**, the solution $x(t)$ of system **1** with the initial value $x(0) = (0, a)^T$ satisfies that, for all $t \geq 0$,

$$|x_j(t)| = |a| \leq C_0 \leq S_j, \quad 1 \leq j \leq n_t,$$

$$|\Delta(x)(2t+1)| = |a| \leq M_0/2,$$

and

$$|\Delta(x)(2t_k+2)| = |a| \left| 1 - \sum_{i=1}^{2t_k+1} b_{2i}(2t_k+1) \right| > M_0.$$

Therefore, the model is a devil network.

To this end, it should be noted that in the second case above, for some initial values the storages of the server (or the whole network) can oscillate more and more strongly as time evolves.

Example. Consider the special case of $B(2t+1) = (B_1(2t+1), 0)$, where 0 is a $2 \times t$ zero matrix and

$$B_1(2t+1) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ -1 & -1 & \cdots & -1 \end{pmatrix}$$

is a $2 \times (t+1)$ T-matrix. It is clear that $B(2t+1)$ satisfies the condition given in Case II above. So, the network is a devil network. It follows from **7** that

$$x(2t+1) = (b, -a, 0)^T, \quad x(2t+2) = (a, b, a, \dots, a, 0)^T, \quad t \in \mathbf{Z}^+,$$

where 0 in the first relation is a $(2t+2)$ -dimensional zero row vector and 0 in the second relation is a $(t+1)$ -dimensional zero row vector. Hence, for all $t \in \mathbf{Z}^+$, one has $|x_j(t)| \leq \max\{|a|, |b|\}$, $1 \leq j \leq n_t$, and

$$|\Delta(x)(2t+1)| = |b-a|, \quad |\Delta(x)(2t+2)| = |b+(t+2)a|. \quad [13]$$

Obviously, in the case of $a \neq 0$, the storage of the server strongly oscillates as time evolves. This illustrates that the network runs quite well at some times, but will break down at some other times, when time is sufficiently large.

Moreover, the matrix $B(2t+1)$ in this example describes the phenomenon that the second computer in the network continuously sends data $a \neq 0$ to each of the other $t+1$ computers. Although the burden of each of these $t+1$ computers received from this second computer is equal to a , which is not heavy if $|a|$ is small, it gives an extra load to the server. If the network is an email system, this example explains why a virus-contaminated computer

can cause the server to breakdown since burden is continuously building up on the server in this way as described by the new model.

Remark 2.

- (1) Since the coupling matrix $A(t)$ of the computer network discussed in this section is a T-matrix, all the entries of $D(t)$ defined by **6** are integers. So, system **1** cannot be chaotic in the sense of Li-Yorke. However, when the connections are weighted or time-varying, the linear system **1** may become chaotic in the sense of Li-Yorke (5), which will be further discussed elsewhere in the near future.
- (2) In the above example, if $y = \Delta(x)$ is taken as an output of the system, the output according to **13** is chaotic in the sense of Li-Yorke. In fact, in this case, there is an uncountable scrambled set in the diagonal line $\{(a, a) : a \in \mathbf{R}\}$ (6).

3. Stability for a general linear model of networks

Consider the stability of a general model of networks, i.e., its corresponding system **1** is linear, in which its connections may be directed and weighted, and its dimension, connectivity as well as weights may vary with time.

Let n_t be the number of all the nodes in the network at time t . Suppose that $x_j(t)$ represents a quantity of some property \mathcal{P} of node j at time t , $1 \leq j \leq n_t$, and $x(t) = (x_1(t), x_2(t), \dots, x_{n_t}(t))^T \in \mathbf{R}^{n_t}$ satisfies the linear system **1**, where $A(t) = (a_{ij}(t))$ is an $n_{t+1} \times n_t$ matrix and its entry $a_{ij}(t)$ represents a weight, which is no longer restricted to the set of $\{-1, 0, 1\}$, with direction from node i to node j at time t .

Similarly assume that each node i in the network has its own maximum quantity (e.g., storage) S_i , invariant in time, and the whole network has its own maximum quantity which may be infinite or varying with time, for the property \mathcal{P} .

The following discussion is divided into two cases: (1) the maximum quantity for property \mathcal{P} of the whole network is time-invariant, which can be either finite or infinite; (2) the maximum quantity for property \mathcal{P} of the whole network is time-varying.

3.1. Networks with time-invariant maximum quantity of property \mathcal{P}

Let M_0 be the maximum quantity for property \mathcal{P} of the whole network, which is a positive constant or infinity. In this case, the definitions of stable, unstable, and devil networks are similar to those given in Definition 1 in Section 2.

It is clear that for any given initial point $x(0) \in \mathbf{R}^{n_0}$, the solution $x(t)$ of system **1** can also be written as **2**, with $D(t) = (d_{ij}(t))_{n_{t+1} \times n_0}$ satisfying **3**.

Next, the stability and instability of system **1** are studied for the case of $M_0 < \infty$.

Theorem 1. Assume that the maximum quantities M_0 and S_i for property \mathcal{P} of the whole network and of each node i are both finite. Then, the network described by system **1** is stable if and only if $\left\{ \sum_{i=1}^{n_{t+1}} d_{ij}(t) \right\}_{t=0}^{\infty}$ is bounded for all $1 \leq j \leq n_0$, and moreover there exists a positive constant β such that

$$|d_{ij}(t)| \leq \beta S_i, \quad 1 \leq j \leq n_0, \quad 1 \leq i \leq n_{t+1}, \quad t \in \mathbf{Z}^+. \quad [14]$$

Proof. First, the sufficiency is verified. By the assumption, there exists a constant $\gamma > 0$ such that

$$\left| \sum_{i=1}^{n_{t+1}} d_{ij}(t) \right| \leq \gamma, \quad 1 \leq j \leq n_0, \quad t \in \mathbf{Z}^+. \quad [15]$$

It follows from **2**, **14**, and **15** that, for all $t \in \mathbf{Z}^+$,

$$|x_i(t+1)| \leq \sum_{j=1}^{n_0} |d_{ij}(t)| |x_j(0)| \leq \beta S_i \sum_{j=1}^{n_0} |x_j(0)|, \quad 1 \leq i \leq n_{t+1},$$

and

$$\begin{aligned} |\Delta(x)(t+1)| &= \left| \sum_{i=1}^{n_{t+1}} \left(\sum_{j=1}^{n_0} d_{ij}(t) x_j(0) \right) \right| \\ &\leq \sum_{j=1}^{n_0} \left| \sum_{i=1}^{n_{t+1}} d_{ij}(t) \right| |x_j(0)| \leq \gamma \sum_{j=1}^{n_0} |x_j(0)|. \end{aligned} \quad [16]$$

So, for any initial point $x(0) \in \mathbf{R}^{n_0}$ with $\sum_{j=1}^{n_0} |x_j(0)| \leq r_0$, where

$$r_0 = \min\{S_1, S_2, \dots, S_{n_0}, 1/\beta, M_0, M_0/\gamma\}, \quad [17]$$

one has

$$|x_i(t)| \leq S_i, \quad 1 \leq i \leq n_t, \quad |\Delta(x)(t)| \leq M_0, \quad t \in \mathbf{Z}^+. \quad [18]$$

Hence, the network is stable.

Then, the necessity is verified. Since the network is stable, there exists a positive constant r_0 such that for any initial point $x(0) \in \mathbf{R}^{n_0}$ with $\sum_{j=1}^{n_0} |x_j(0)| \leq r_0$ one has $|x_i(t+1)| \leq$

S_i , $1 \leq i \leq n_{t+1}$, and $|\Delta(x)(t+1)| \leq M_0$, for all $t \in \mathbf{Z}^+$. Given any j_0 , $1 \leq j_0 \leq n_0$, set $x_{j_0}(0) = r_0$ and $x_j(0) = 0$, $1 \leq j \neq j_0 \leq n_0$. Then, it follows from **2** that, for all $t \in \mathbf{Z}^+$,

$$|x_i(t+1)| = |d_{ij_0}(t)x_{j_0}(0)| = r_0 |d_{ij_0}(t)| \leq S_i, \quad 1 \leq i \leq n_{t+1},$$

$$|\Delta(x)(t+1)| = \left| \sum_{i=1}^{n_{t+1}} d_{ij_0}(t)x_{j_0}(0) \right| = r_0 \left| \sum_{i=1}^{n_{t+1}} d_{ij_0}(t) \right| \leq M_0,$$

which implies that, for all $t \in \mathbf{Z}^+$,

$$|d_{ij_0}(t)| \leq S_i/r_0, \quad 1 \leq i \leq n_{t+1}, \quad \left| \sum_{i=1}^{n_{t+1}} d_{ij_0}(t) \right| \leq M_0/r_0.$$

Hence, inequality 14 holds with $\beta = 1/r_0$, and $\left\{ \sum_{i=1}^{n_{t+1}} d_{ij}(t) \right\}_{t=0}^{\infty}$ is bounded for all $1 \leq j \leq n_0$. The necessity is thus verified.

Therefore, the proof is complete.

Theorem 2. Assume that the maximum quantities M_0 and S_i for property \mathcal{P} of a whole network and each node i are both finite. Then, the network described by system **1** is a devil network if there exists a positive constant β such that

$$|d_{ij}(t)| \leq \beta S_i, \quad 1 \leq j \leq n_0, \quad 1 \leq i \leq n_{t+1}, \quad t \in \mathbf{Z}^+,$$

and moreover there exist two time subsequences, $\{t_k\}_{k=1}^{\infty}$ and $\{t'_k\}_{k=1}^{\infty}$ with $t_k \rightarrow \infty$ and $t'_k \rightarrow \infty$ as $k \rightarrow \infty$, such that $\left\{ \sum_{i=1}^{n_{t_k+1}} d_{ij}(t_k) \right\}_{k=1}^{\infty}$ is bounded for all $1 \leq j \leq n_0$ and $\left\{ \sum_{i=1}^{n_{t'_k+1}} d_{ij_0}(t'_k) \right\}_{k=1}^{\infty}$ is unbounded for some $1 \leq j_0 \leq n_0$.

Proof. Since $\left\{ \sum_{i=1}^{n_{t_k+1}} d_{ij}(t_k) \right\}_{k=1}^{\infty}$ is bounded for all $1 \leq j \leq n_0$, there exists a constant $\gamma > 0$ such that

$$\left| \sum_{i=1}^{n_{t_k+1}} d_{ij}(t_k) \right| \leq \gamma, \quad k \geq 1, \quad 1 \leq j \leq n_0.$$

By assumption, $\left\{ \sum_{i=1}^{n_{t'_k+1}} d_{ij_0}(t'_k) \right\}_{k=1}^{\infty}$ is unbounded for some $1 \leq j_0 \leq n_0$. Without loss of generality, suppose that

$$\left| \sum_{i=1}^{n_{t'_k+1}} d_{ij_0}(t'_k) \right| \rightarrow \infty \quad \text{as } k \rightarrow \infty.$$

Then, for any positive constant $r \leq \min\{S_1, S_2, \dots, S_{n_0}, 1/\beta, M_0, \alpha M_0/\gamma\}$, where α , $0 < \alpha < 1$, is the scaling parameter for system **1**, there exists $k_0 \geq 1$ such that

$$\left| \sum_{i=1}^{n_{t'_k+1}} d_{ij_0}(t'_k) \right| > M_0/r, \quad k \geq k_0.$$

Choose an initial point $x(0) \in \mathbf{R}^{n_0}$ with $x_{j_0}(0) = r$ and $x_j(0) = 0$ for all $1 \leq j \neq j_0 \leq n_0$. It is clear that $\sum_{j=1}^{n_0} |x_j(0)| = r$. With an argument similar to that used in the proof of the sufficiency of Theorem 1, one can easily show that the corresponding solution $x(t)$ satisfies

$$|x_i(t)| \leq S_i, \quad 1 \leq i \leq n_t, \quad t \in \mathbf{Z}^+,$$

$$|\Delta(x)(t_k + 1)| \leq \alpha M_0, \quad k \geq 1, \quad |\Delta(x)(t'_k + 1)| > M_0, \quad k \geq k_0.$$

Therefore, the network described by system **1** is a devil network. This completes the proof.

In the case of $M_0 = \infty$, the following result can be easily verified by an argument similar to that used in the proof of Theorem 1.

Theorem 3. Assume that the maximum quantity S_i of property \mathcal{P} for each node i in a network is finite and the maximum quantity for property \mathcal{P} of the whole network is infinite. Then, the network described by system **1** is stable if and only if there exists a constant $\beta > 0$ such that

$$|d_{ij}(t)| \leq \beta S_i, \quad 1 \leq j \leq n_0, \quad 1 \leq i \leq n_{t+1}, \quad t \in \mathbf{Z}^+.$$

The model discussed in Section 2 is revisited here based on the results obtained above. In this model, $n_0 = 2$, and it follows from **6** that, for all $t \in \mathbf{Z}^+$,

$$|d_{ij}(t)| \leq 1 \leq S_i/C_0, \quad 1 \leq i \leq t+3, \quad j = 1, 2,$$

$$\sum_{i=1}^{2t+3} d_{i1}(2t) = 1, \quad \sum_{i=1}^{2t+3} d_{i2}(2t) = -1,$$

$$\sum_{i=1}^{2t+4} d_{i1}(2t+1) = 1 - \sum_{i=1}^{2t+1} b_{2i}(2t+1), \quad \sum_{i=1}^{2t+4} d_{i2}(2t+1) = 1 + \sum_{i=1}^{2t+1} b_{1i}(2t+1),$$

where C_0 is defined by **11**. Therefore, by Theorem 1, this network is stable if and only if the two sequences $\left\{ \sum_{i=1}^{2t+1} b_{1i}(2t+1) \right\}_{t=0}^{\infty}$ and $\left\{ \sum_{i=1}^{2t+1} b_{2i}(2t+1) \right\}_{t=0}^{\infty}$ are bounded. Further, by Theorem 2, this network is a devil network if and only if at least one of the two sequences, $\left\{ \sum_{i=1}^{2t+1} b_{1i}(2t+1) \right\}_{t=0}^{\infty}$ and $\left\{ \sum_{i=1}^{2t+1} b_{2i}(2t+1) \right\}_{t=0}^{\infty}$, is unbounded. These conclusions are the same as those obtained in Section 2.

3.2. Networks with time-varying maximum quantity for property \mathcal{P}

Let $M(t)$ be a maximum quantity of property \mathcal{P} for the whole network at time t .

In this case, the definitions of stable and unstable networks are also similar to those given in Definition 1. For convenience, they are rephrased as follows.

Definition 2. A network is said to be stable if there exists a positive constant r_0 such that for all initial points $x(0) = (x_1(0), x_2(0), \dots, x_{n_0})^T \in \mathbf{R}^{n_0}$ with $\sum_{i=1}^{n_0} |x_i(0)| \leq r_0$, one has $|x_i(t)| \leq S_i$, $1 \leq i \leq n_t$, and $|\Delta(x)(t)| \leq M(t)$ for all $t \in \mathbf{Z}^+$. Otherwise, it is said to be unstable. In particular, a network is called a devil network if it is unstable and, moreover, for any small positive constant r there exists an initial point $x(0) \in \mathbf{R}^{n_0}$ with $\sum_{i=1}^{n_0} |x_i(0)| < r$ such that $|x_i(t)| \leq S_i$, $1 \leq i \leq n_t$, for all $t \in \mathbf{Z}^+$, $|\Delta(x)(t_k)| \leq \alpha M(t_k)$ for infinitely many times t_k , for some constant $0 < \alpha < 1$, and $|\Delta(x)(t'_k)| > M(t'_k)$ for infinitely many times t'_k , $k \geq 1$.

Theorem 4. Assume that the maximum quantity S_i of property \mathcal{P} for each node i in the network is finite, and the maximum quantity $M(t)$ of property \mathcal{P} for the whole network is finite, at any time $t \in \mathbf{Z}^+$. Then, the network described by system **1** is stable if and only if there exist positive constants β and γ such that

$$|d_{ij}(t)| \leq \beta S_i, \quad 1 \leq i \leq n_{t+1}, \quad \left| \sum_{i=1}^{n_{t+1}} d_{ij}(t) \right| \leq \gamma M(t+1), \quad 1 \leq j \leq n_0, \quad t \in \mathbf{Z}^+.$$

Proof. The proof of the theorem is similar to that of Theorem 1, except the following: (i) in the proof of the sufficiency, γ in **15** and **16** is replaced by $\gamma M(t+1)$, r_0 in **17** is replaced by

$$r_0 = \min\{S_1, S_2, \dots, S_{n_0}, M(0), 1/\beta, 1/\gamma\},$$

and M_0 in **18** is replaced by $M(t)$; (ii) in the proof of the necessity, M_0 is replaced by $M(t+1)$.

This completes the proof.

Similar to Theorem 2, the following result can be established.

Theorem 5. Assume that the maximum quantity S_i of property \mathcal{P} for each node i in the network is finite, and a maximum quantity $M(t)$ of property \mathcal{P} for the whole network is finite, at any time $t \in \mathbf{Z}^+$. Then, the network described by system **1** is a devil network if there exists a positive constant β such that

$$|d_{ij}(t)| \leq \beta S_i, \quad 1 \leq j \leq n_0, \quad 1 \leq i \leq n_{t+1}, \quad t \in \mathbf{Z}^+,$$

and moreover there exist two time subsequences, $\{t_k\}_{k=1}^\infty$ and $\{t'_k\}_{k=1}^\infty$ with $t_k \rightarrow \infty$ and $t'_k \rightarrow \infty$ as $k \rightarrow \infty$, such that $\left\{ M^{-1}(t_k + 1) \sum_{i=1}^{n_{t_k+1}} d_{ij}(t_k) \right\}_{k=1}^\infty$ is bounded for all $1 \leq j \leq n_0$ and $\left\{ M^{-1}(t'_k + 1) \sum_{i=1}^{n_{t'_k+1}} d_{ij_0}(t'_k) \right\}_{k=1}^\infty$ is unbounded for some $1 \leq j_0 \leq n_0$.

Proof. The proof of the theorem is similar to that of Theorem 2. So the details are omitted.

Remark 3.

- (1) If the number of nodes in a network does not vary; that is, the dimension n_t of the corresponding system **1** is time-invariant, then system **1** is a classical time-varying discrete linear system, which surprisingly can be chaotic in the sense of Li-Yorke (5).
- (2) If the effects of internal and external nonlinearities on a network are considered, the corresponding system is by nature nonlinear, for which the stability and complex dynamical behaviors need to be further addressed in the future.

Remark 4. In the present letter, only the simplest possible model of a localized isolated computer network with one server is considered. A realistic computer model, however, has more than one server in general, which becomes more mathematically involved, leaving a challenging topic for future research.

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